A SAT-Based Approach

for Solving the Minimal S5-Satisfiability Problem

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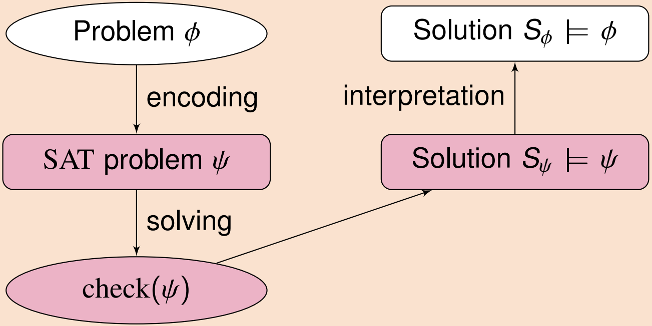
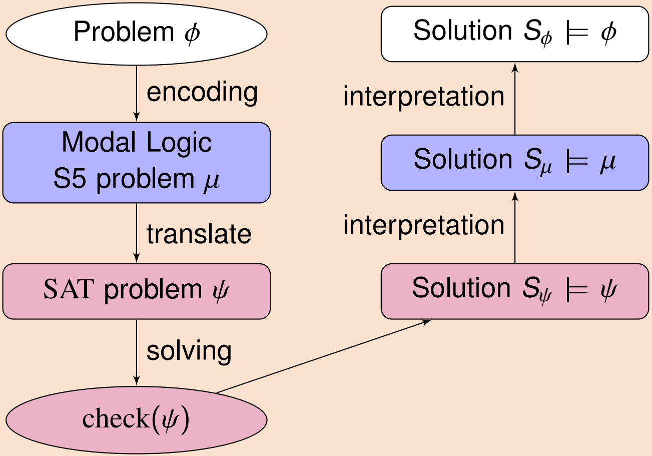
For many **NP** problems, here the current most efficient technique:

SAT solving



Introduction: S5-Satisfiability Problem as SAT

S52SAT solving



Introduction: Why we want to minimize ?

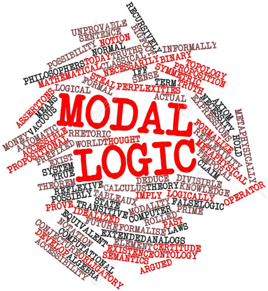
Motivations

- Modal Logic S5 is more expressive than propositional logic
- They have the same complexity (**NP**-complete)
- Modal logic S5 models are large (roughly one variable assignment per world)
- For some polynomial reduction to ML-S5, we may want a small model

Question: How to obtain the smallest model in number of worlds for modal logic S5 ?

Modal Logic S5

- $\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Box\phi \mid \Diamond\phi$
- $\Box\phi$ means ϕ is necessarily true
- $\Diamond\phi$ means ϕ is possibly true
- There exists a lot of different modal logics
- Most of them have their satisfiability in **PSPACE**



Satisfiability in Modal Logic S5 is **NP**-Complete [Lad77]

Preliminaries: S5-Structure

- $\mathbb{P}$ finite non-empty set of propositional variables

S5-Structure $\mathcal{K}$ [Kri63]

$M = \langle W, R, V \rangle$ with:

- W, a non-empty set of possible worlds
- R, a binary relation on W (which is total: $\forall w.\forall v.\ (w, v) \in R$)
- V, a function that associate to each $p \in \mathbb{P}$, the set of possible worlds where p is true

Pointed S5-Structure: $\langle \mathcal{K}, w \rangle$

- $\mathcal{K}$: S5-Structure
- w: a possible world in W

Preliminaries: S5-Satisfiability

Definition (Satisfaction Relation)

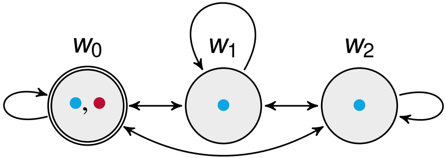
The relation $\models$ between S5-Structures and formulae is recursively defined as follows:

$$
\begin{array}{lll}
\langle \mathcal{K}, w \rangle \models p & \text{iff} & w \in V(p) \\
\langle \mathcal{K}, w \rangle \models \neg\phi & \text{iff} & \langle \mathcal{K}, w \rangle \not\models \phi \\
\langle \mathcal{K}, w \rangle \models \phi_1 \wedge \phi_2 & \text{iff} & \langle \mathcal{K}, w \rangle \models \phi_1 \text{ and } \langle \mathcal{K}, w \rangle \models \phi_2 \\
\langle \mathcal{K}, w \rangle \models \phi_1 \vee \phi_2 & \text{iff} & \langle \mathcal{K}, w \rangle \models \phi_1 \text{ or } \langle \mathcal{K}, w \rangle \models \phi_2 \\
\langle \mathcal{K}, w \rangle \models \Box\phi & \text{iff} & (w, w') \in R \text{ implies } \langle \mathcal{K}, w' \rangle \models \phi \\
\langle \mathcal{K}, w \rangle \models \Diamond\phi & \text{iff} & (w, w') \in R \text{ and } \langle \mathcal{K}, w' \rangle \models \phi
\end{array}
$$

$\mathcal{K}$ that satisfied a formula ϕ will be called “model of ϕ”

Preliminaries: S5-Satisfiability

- ✓ $\phi_1 = \Box(\bullet)$
- ✗ $\phi_2 = \Box\Diamond(\bullet)$
- ✓ $\phi_3 = (\bullet \vee \bullet)$
- ✓ $\phi_4 = \Diamond(\bullet \wedge \Box\neg\bullet)$



Preliminaries: Upper-bound on $\mathcal{K}$

ϕ is considered in NNF (negation only on variables)

Definition (Diamond-Degree)

$$\begin{aligned}
dd(\top) &= dd(\neg\top) = 0 \\
dd(p) &= dd(\neg p) = 0 \\
dd(\psi \wedge \chi) &= dd(\psi) + dd(\chi) \\
dd(\psi \vee \chi) &= \max(dd(\psi), dd'(\chi)) \\
dd(\Box\psi) &= dd(\psi) \\
dd(\Diamond\psi) &= 1 + dd(\psi)
\end{aligned}$$

For any $\mathcal{K}$ satisfying a formula ϕ in S5, $|\mathcal{K}| \leq dd(\phi) + 1$

From Modal Logic to SAT

$$\mathrm{tr}(\phi, n) = \mathrm{tr}'(\phi, 1, n)$$

$$\mathrm{tr}'(p, i, n) = p_i \quad \mathrm{tr}'(\neg p, i, n) = \neg p_i$$

$$\mathrm{tr}'(\psi \wedge \chi, i, n) = \mathrm{tr}'(\psi, i, n) \wedge \mathrm{tr}'(\chi, i, n)$$

$$\mathrm{tr}'(\psi \vee \chi, i, n) = \mathrm{tr}'(\psi, i, n) \vee \mathrm{tr}'(\chi, i, n)$$

$$\mathrm{tr}'(\Box\psi, i, n) = \bigwedge_{j=1}^{n}(\mathrm{tr}'(\psi, j, n))$$

$$\mathrm{tr}'(\Diamond\psi, i, n) = \bigvee_{j=1}^{n}(\mathrm{tr}'(\psi, j, n))$$

S52SAT = $\mathrm{tr}(\mathrm{nnf}(\phi), \mathrm{dd}(\phi) + 1)$

Preliminaries: Goal

What we want

- We know how to find a $\mathcal{K}$ that satisfies a formula
- We know that an upper-bound for $|\mathcal{K}|$ is dd$(\phi) + 1$ [CLL+17]
- The solver S52SAT translates ϕ into SAT to find it [CLL+17]
- S52SAT always find a model of size dd$(\phi) + 1$ by construction

How to find the smallest $\mathcal{K}$ that satisfied a formula ?

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Preliminaries: Goal

What we want

- We know how to find a $\mathcal{K}$ that satisfies a formula
- We know that an upper-bound for $|\mathcal{K}|$ is $\mathrm{dd}(\phi) + 1$ [CLL+17]
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How to find the smallest $\mathcal{K}$ that satisfied a formula ?

Many different techniques that we will discuss here:

- MaxSAT solvers
- MCS Extraction
- Unsatisfiable core in S52SAT

Contribution: additional reasoning using selectors

Translation with selectors on conjunctions and disjunctions

$$\mathrm{tr_{sel}}(\phi, n) = \mathrm{tr_{sel}}'(\phi, 1, n)$$
$$\mathrm{tr_{sel}}'(p, i, n) = p_i$$
$$\mathrm{tr_{sel}}'(\neg p, i, n) = \neg p_i$$
$$\mathrm{tr_{sel}}'(\psi \wedge \chi, i, n) = \mathrm{tr_{sel}}'(\psi, i, n) \wedge \mathrm{tr_{sel}}'(\chi, i, n)$$
$$\mathrm{tr_{sel}}'(\psi \vee \chi, i, n) = \mathrm{tr_{sel}}'(\psi, i, n) \vee \mathrm{tr_{sel}}'(\chi, i, n)$$
$$\mathrm{tr_{sel}}'(\Box\psi, i, n) = \bigwedge_{j=1}^{n}(\neg s_j \vee (\mathrm{tr_{sel}}'(\psi, j, n)))$$
$$\mathrm{tr_{sel}}'(\Diamond\psi, i, n) = \bigvee_{j=1}^{n}(s_j \wedge (\mathrm{tr_{sel}}'(\psi, j, n)))$$

Contributions: Important lemmas for the approach

Lemma 1

If $\mathrm{tr}_{\mathrm{sel}}(\phi, n)$ is unsatisfiable and the solver returns an unsatisfiable core of size m, then all groups of m selectors are also an unsatisfiable core.

Lemma 2

If $\mathrm{tr}_{\mathrm{sel}}(\phi, n)$ is unsatisfiable and the solver returns an unsatisfiable core of size m then $\forall n' \in \{1, \ldots, (\mathrm{dd}(\phi) - m)\}$, $\mathrm{tr}_{\mathrm{sel}}(\phi, n')$ is unsatisfiable.

Contributions: S52SAT $1toN_c$

Data: ϕ, a modal logic formula
Result: $\langle M, w\rangle$ such that $\langle M, w\rangle \models_{min} \phi$, else UNSAT

```
begin
    b ← 1 ;
    n ← dd(ϕ) + 1 ;
    ⟨r, s⟩ ← glucose(tr_sel(ϕ, n), b) ;
    while (r ≠ SAT ∧ (b ≤ n)) do
        b ← b + (n − |s| + 1) ;
        ⟨r, s⟩ ← glucose(tr_sel(ϕ, n), b) ;
    end
    if r ≠ SAT then
        return UNSAT ;
    else
        ⟨M, w⟩ ← getS5Model(s) ;
        return ⟨M, w⟩;
    end
end
```

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Experimental settings

Settings

- Cluster of Xeon 4 cores, 3.3 GHz, running CentOS 6.4
- Memory limit: 32 GB
- Time limit: 900 seconds (per solver per benchmark)

Benchmarks

- Random QBF (MQBF) formulae [MD00, KT13]
- Random 3CNF formulae of modal depths 1 and 2 [PS03]
- Crafted LWB K, KT and S4 formulae [BHS00]

Experimental settings

Solvers

- S52SAT [CLL+17] + 6 different strategies:
  - $1toN_c$ - 1toN
  - $Nto1_c$ - Nto1
  - $Dicho_c$ - Dicho
- CNF with soft clauses with MaxSAT solvers:
  - maxHS-b [DB11]
  - mscg2015b [dRMIS14]
  - MSUnCore [HMMS11]
- MCS extraction with the LBX solver [MPM15]

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Experiments on 3CNF

Method	md=1	md=2	Total
1toN	55 / 102	17 / 355	72 / 457
Nto1	0 / 102	0 / 355	0 / 457
Dicho	26 / 102	9 / 355	35 / 457
$1toN_c$	**62** / 102	**27** / 355	**89** / 457
$Nto1_c$	0 / 102	0 / 355	0 / 457
$Dicho_c$	40 / 102	17 / 355	57 / 457
maxHS	49 / 102	17 / 355	66 / 457
MSCG	31 / 102	17 / 355	48 / 457
MSUnCore	40 / 102	17 / 355	57 / 457
LBX	47 / 102	17 / 355	64 / 457
VBS	62 / 102	27 / 355	89 / 457

Table: #instances solved in 3*CNF*

Experiments on MQBF

Method	qbf	qbfL	qbfS	qbfMS	Total
1toN	**56** / 56	0 / 240	**171** / 240	0 / 240	**227** / 617
Nto1	55 / 56	0 / 240	0 / 240	0 / 240	55 / 617
Dicho	**56** / 56	0 / 240	**171** / 240	0 / 240	**227** / 617
$1toN_c$	**56** / 56	0 / 240	**171** / 240	0 / 240	**227** / 617
$Nto1_c$	50 / 56	0 / 240	0 / 240	0 / 240	50 / 617
$Dicho_c$	**56** / 56	0 / 240	**171** / 240	0 / 240	**227** / 617
maxHS	**56** / 56	0 / 240	0 / 240	0 / 240	56 / 617
MSCG	**56** / 56	0 / 240	**171** / 240	0 / 240	**227** / 617
MSUnCore	**56** / 56	0 / 240	0 / 240	0 / 240	56 / 617
LBX	**56** / 56	0 / 240	167 / 240	0 / 240	223 / 617
VBS	56 / 56	0 / 240	171 / 240	0 / 240	227 / 617

Table: #instances solved in *MQBF*

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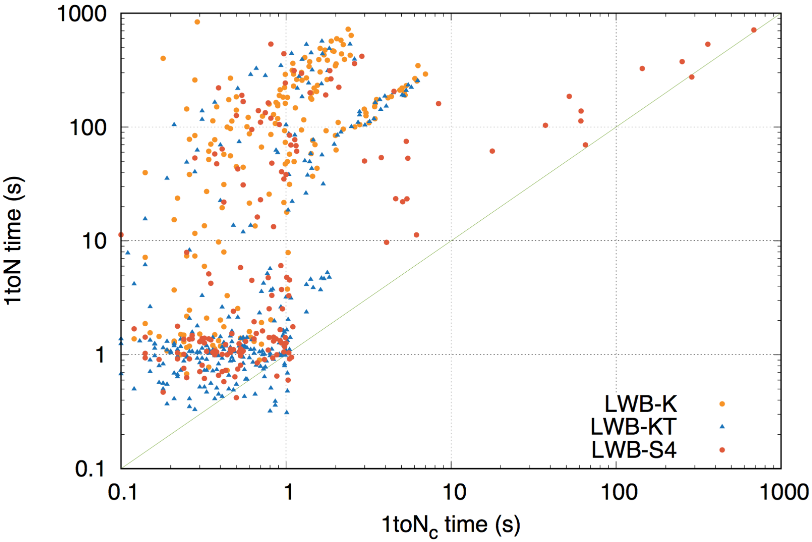
Results on LWB

Method	LWB K	LWB KT	LWB S4	Total
1toN	**185** / 504	**279** / 504	**160** / 504	**624** / 1512
Nto1	17 / 504	34 / 504	2 / 504	53 / 1512
Dicho	119 / 504	175 / 504	78 / 504	372 / 1512
$1toN_c$	**185** / 504	**279** / 504	**160** / 504	**624** / 1512
$Nto1_c$	17 / 504	33 / 504	0 / 504	50 / 1512
$Dicho_c$	135 / 504	201 / 504	100 / 504	436 / 1512
maxHS	21 / 504	42 / 504	82 / 504	145 / 1512
MSCG	121 / 504	105 / 504	139 / 504	365 / 1512
MSUnCore	21 / 504	42 / 504	82 / 504	145 / 1512
LBX	118 / 504	173 / 504	92 / 504	383 / 1512
VBS	185 / 504	279 / 504	160 / 504	624 / 1512

Table: #instances solved in LWB

Impact of the additional reasoning with selectors (LWB)



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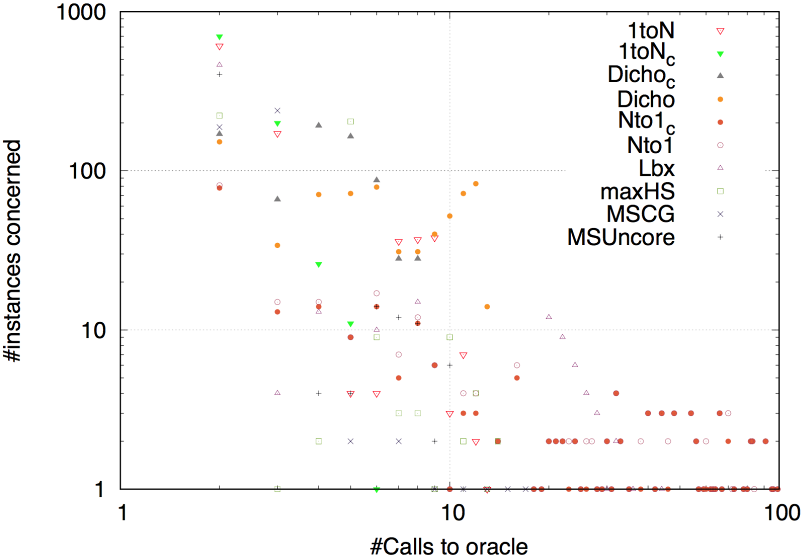
Comparison of the number of calls to the SAT oracle



Conclusion & Perspectives

Conclusion

- We defined the problem MinS5-SAT
- Different techniques to solve it
- An elegant way to use selectors to minimize a model
- Empirically, formulae in different modal logics are sometimes S5-satisfiable

Conclusion & Perspectives

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Perspectives

- Use S52SAT 1toN_c as preprocessor for modal logic K
- Try to minimize Kripke model for other modal logics
- Find practical applications that can be translated into modal logic S5

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